

Curling Numbers of Certain Graph Powers

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Abstract

Given a finite nonempty sequence S of integers, write it as XY^k , where Y^k is a power of greatest exponent that is a suffix of S : this k is the curling number of S . The concept of curling number of sequences has already been extended to the degree sequences of graphs to define the curling number of a graph. In this paper we study the curling number of graph powers, graph products and certain other graph operations.

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1 Introduction

For terms and definitions in graph theory, we refer to [2, 4, 5, 7, 9] and for more about different graph classes we refer to [3, 6]. All graphs we use here are simple, finite, connected and undirected. The notion of curling number of integer sequences are introduced in [1] as follows.

Definition 1.1. [1] Given a finite nonempty sequence S of integers, write it as XY^k , where Y^k is a power of greatest exponent that is a suffix of S : this k is the curling number of S .

The concept of curling number of integer sequences has been extended to the degree sequences of graphs in [8] and the corresponding properties and characteristics of certain standard graphs have been studied in that paper.

Some of the main results on curling numbers are follows.

Definition 1.2. The *Curling Number Conjecture* (see [1]) states that if one starts with any finite string, over any alphabet, and repeatedly extends it by appending the curling number of the current string, then eventually one must reach a 1.

Definition 1.3. [8] A maximal degree subsequence with equal entries is called an *identity subsequence*. An identity subsequence can be a curling subsequence and the number of identity curling subsequences found in a simple connected graph G is denoted $ic(G)$

Theorem 1.4. [8] *The number of curling subsequences of a simple connected graph G is given by*

$$\vartheta(G) = \begin{cases} 1 & ; \text{ if } ic(G) = 1; \\ ic(G)+ic(G)! & ; \text{ otherwise.} \end{cases}$$

Theorem 1.5. [8] *For the degree sequence of a non-trivial, connected graph G on n vertices, the curling number conjecture holds.*

Theorem 1.6. [8] *If a graph G is the union of m simple connected graphs G_i ; $1 \leq i \leq m$ and the respective degree sequences are re-arranged as strings of identity subsequences, then*

$$cn(G) = \begin{cases} \max\{cn(G_i)\} & ; \text{ if } X_i, X_j \text{ are not pairwise similar,} \\ \max\sum_{i=1}^m k_i & ; \text{ for all integer of similar identify subsequences.} \end{cases}$$

As stated earlier, any degree sequence of a graph G can be written as a string of identity curling subsequences. In view of this fact, the concept of compound curling number of a graph G has been introduced in [8] as follows.

Definition 1.7. Let the degree sequence of the graph G be written as a string of identity curling subsequences say, $X_1^{k_1} \circ X_2^{k_2} \circ X_3^{k_3} \dots \circ X_l^{k_l}$. The *Compound curling number* of G , denoted $cn^c(G)$ is defined to be,

$$cn^c(G) = \prod_{i=1}^l k_i.$$

The curling number and compound curling number of certain fundamental standard graphs have been determined in [8] and the major results are listed in the following table.

Sl.No.	Graph	cn	cn^c
1	Complete Graph $K_n, n \geq 1$	n	n
2	Complete Bipartite Graph $K_{m,n}, m \neq n$	$\max \{m, n\}$	mn
3	Complete Bipartite Graph $K_{n,n}$	$2n$	n^2
4	Path Graph $P_n, n \geq 3$	$n - 2$	$2(n - 2)$
5	Cycle C_n	n	n
6	Wheel Graph $W_n = C_{n-1} + K_1$	$n - 1$	$n - 1$
7	Ladder Graph $L_n = P_n \times P_2, n \geq 2$	$2(n - 2)$	$8(n - 2)$

Proposition 1.8. [8] *The compound curling number of any regular graph G is equal to its curling number.*

2 New Results

In this paper, we extend these studies on curling number to the integral powers of certain graph classes. By the size of a sequence, we mean the number of elements in that sequence.

2.1 Curling number of certain graph powers

Let us first recall the definition of integer powers of graphs as follows.

Definition 2.1. [2] The r -th power of a simple graph G is the graph G^r whose vertex set is V , two distinct vertices being adjacent in G^r if and only if their distance in G is at most r . The graph G^2 is referred to as the square of G , the graph G^3 as the cube of G .

The following is an important theorem on graph powers.

Theorem 2.2. [10] *If d is the diameter of a graph G , then G^d is a complete graph.*

If d is the diameter of a graph G , then for any integer $q \geq d$, it can be noted that G^q is a complete graph. Therefore we need to consider an integer $r \leq d$ to construct the r -th power of a given graph G .

From the table in the above section, it can be seen that the curling number of a complete graph is equal to the order of it. Therefore for any graph G of diameter d , we have $cn(G^r) = |V(G)|$, where $r \geq d$. Hence, we consider an integer r , where $1 \leq r \leq d$, as the power of a given graph G in our present study.

In the following theorem we determine the curling number of the powers of path graphs.

Theorem 2.3. *For $n \geq 3$, let P_n be a path on n vertices and let $r \leq n - 1$ be a positive integer, then the curling number of the r th power of G is given by*

$$cn(P_n^r) = \begin{cases} 2 & ; \text{if } r = \lfloor \frac{n}{2} \rfloor; \\ n - 2r & ; \text{if } r < \lfloor \frac{n}{2} \rfloor - 1; \\ 2(r + 1) - n & ; \text{if } \lfloor \frac{n}{2} \rfloor \leq r \leq n - 1. \end{cases}$$

Proof. Let P_n be a path on n vertices and r be a positive integer less than or equal to n . Here we have to consider the following two cases.

Case - 1: Assume that $r \leq \lfloor \frac{n}{2} \rfloor$. The degree sequence of P_n^r is $(r, r + 1, r + 2, \dots, 2r - 1, \underbrace{2r, 2r, \dots, 2r}_{(n-2r) \text{ terms}}, 2r - 1, \dots, r + 2, r + 1, r)$. Therefore, the degree sequence becomes

$(r, r + 1, r + 2, r + 3, \dots, 2r - 1)^2 \circ (2r)^{n-2r}$. If $n - 2r < 2$, the curling number of $P_n^r = 2$ and if $n - 2r \geq 2$, then the curling number of P_n^r is $n - 2r$.

Case - 2: Assume that $r > \lfloor \frac{n}{2} \rfloor$. The degree sequence of P_n^r is $(r, r + 1, r + 2, \dots, 2r - 3, \underbrace{2r - 2, 2r - 2, \dots, 2r - 2}_{2(r+1)-n \text{ terms}}, 2r - 3, \dots, r + 2, r + 1, r)$. Therefore, the degree sequence

becomes $(r, r + 1, r + 2, r + 3, \dots, 2r - 3)^2 \circ (2r - 2)^{2(r+1)-n}$. That is, in this case, the curling number of P_n^r is $2(r + 1) - n$. Therefore, we have

$$cn(P_n^r) = \begin{cases} 2 & ; \text{if } r = \lfloor \frac{n}{2} \rfloor; \\ n - 2r & ; \text{if } r < \lfloor \frac{n}{2} \rfloor - 1; \\ 2(r + 1) - n & ; \text{if } \lfloor \frac{n}{2} \rfloor \leq r \leq n - 1. \end{cases}$$

This completes the proof. \square

Corollary 2.4. *Let P_n be a path on n vertices and let $r \leq n$ be a positive integer, then the compound curling number of the r th power of G is given by*

$$cn^c(P_n^r) = \begin{cases} 2^r & ; \text{if } r = \lfloor \frac{n}{2} \rfloor; \\ 2^r(n - 2r) & ; \text{if } r < \lfloor \frac{n}{2} \rfloor - 1; \\ 2^{r-1}(r + 1) - n & ; \text{if } \lfloor \frac{n}{2} \rfloor \leq r \leq n - 1. \end{cases}$$

Proof. The proof of the result is immediate from the degree sequence, as explained in the above theorem. \square

Proposition 2.5. *The curling number is invariant under the r -th power of a cycle C_n , for all r such that $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. For $2 \leq r \leq \lfloor \frac{n}{2} \rfloor$, we can see that the r th power of any cycle C_n is $2r$ -regular graph on n vertices and hence, the graph C_n^r has the degree sequence $(2r)^n$. Therefore, $cn(C_n^r) = n = cn(C_n)$, for any positive integer $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. \square

Another interesting graph class we consider here is the class of tadpole graphs, which are combinations of paths and cycles. The (m, n) - *tadpole graph*, also called a dragon graph, is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge. The curling number of different powers of tadpole graphs are determined in the following theorem.

Theorem 2.6. *For a tadpole graph $G = T_{m,n}$, $cn(G^r) = m + n - 2(2r - 1)$.*

Proof. Let $U_0^{(r)}$ be the maximal identity sequence in the r th power of the tadpole graph G . Clearly, the curling number of G^r is the number of elements in the identity sequence $U_0^{(r)}$. If $r=1$, for a tadpole graph $G = T_{m,n}$, we have the degree sequence is $(3)^1 \circ (1)^1 \circ (2)^{m+n-2}$.

In each case, we need to analyse the size of the maximal identity sequence $U_0^{(r)}$. When r increases by 1, it can be noted that the following changes in the degree sequence. Two vertices of the cycle C_m and one vertex of P_n , other than the common vertex v come out from $U_0^{(r)}$. Also, one end vertex of P_n , other than the end vertex comes out of $U_0^{(r)}$.

Therefore, the total number of vertices that are excluded from the maximal identity sequence $U_0^{(r)}$ will be $(2(r - 1) + 1) + (2r - 1) = 2(2r - 1)$. We can see that all other vertices of G^r will have the same degree $2r$. Therefore, the number of vertices in G^r with degree $2r$ is $m + n - 2(2r - 1)$. Therefore the curling number, $cn(G^r)$ of G^r of a tadpole graph $T_{m,n}$ is $m + n - 2(2r - 1)$. \square

The following result provides the compound curling number of a tadpole graph.

Corollary 2.7. *For a tadpole graph $G = T_{m,n}$, the compound curling number, $cn^c(G^r) = r(r - 1)(m + n - 2(2r - 1))$.*

Proof. The degree sequence of G^r is of the form $(3)^{r-1} \circ (1)^r \circ (2r)^{m+n-2(2r-1)}$. Therefore the compound curling number, $cn^c(G^r) = r(r - 1)(m + n - 2(2r - 1))$. \square

Another fundamental graph structure that arouses much interest for our study in this context is a tree. Since the adjacency and incidence patterns of various trees are uncertain, and hence a study on the curling number of arbitrary trees is very complex, we proceed with trees having specific patterns.

At first, we study the curling number of complete binary trees in the following theorem.

Theorem 2.8. *The curling number of any integral power of a complete binary tree of height $h \geq 2$ is 2^h . Also, the compound curling number of any integral power of a complete binary tree, $cn^c(G^r)$ is $2^{(h+1)C_2}$.*

Proof. A complete binary tree of height h has $2^{h+1} - 1$ vertices such that the root vertex has degree 2, the internal vertices have degree 3 and external vertices have degree 1. On taking higher powers, we notice that the degree of the vertices at the same level have the same degree. Let $r_i, 0 \leq i \leq h$ be the degree of vertices at the i th level. Therefore the degree sequence of the complete binary tree G can be written in the string form as $\prod_{i=0}^h r_i^{2^i}$. Therefore the curling number of G^r is 2^h . It is also clear from the above expression that the compound curling number, $cn^c(G^r) = \prod_{i=0}^h 2^i = 2^{(h+1)C_2}$, where nC_r is the binomial coefficient. \square

As a generalisation of the above theorem, the curling number of a complete n -ary tree is determined as given in the following result.

Theorem 2.9. *The curling number of integral powers of a complete n -ary tree of height h is n^h and the compound curling number of integral powers of a complete n -ary tree of height h is $\prod_{i=0}^h n^i$.*

The proof of the theorem is immediate from the above theorem by taking n in place of 2.

A Caterpillar is a particular type of tree, the study of whose curling numbers seems to be promising in this context. A *caterpillar* is an acyclic graph which reduces to a path on removing its end vertices. The Caterpillar G can be considered as the corona $P_n \odot K_1$. A result on the curling number of an arbitrary caterpillar is described as follows.

Lemma 2.10. *For a caterpillar graph G , $cn(G) = \max\{\eta, \sum_{i=0}^n l_i\}$ where η is the maximum number of times a positive integer appears as a degree of internal vertices of G .*

Proof. Let $V_1 = u_i : 1 \leq i \leq n$ is the set of internal vertices of G and let V_2 be the set of all end vertices of G . Let S_0 be the degree sequence of the vertices in V_2 of G and let η be the maximum number of times a particular number that appears as a degree of vertices in V_1 . For $1 \leq i \leq n$, let l_i be the number of end vertices adjacent to a vertex u_i in $V_1(G)$. Therefore the degree sequence of the vertices in V_2 can be written as $(1)^{\sum_{i=0}^n l_i}$. Therefore clearly the curling number of G , $cn(G) = \max\{\eta, \sum_{i=0}^n l_i\}$. \square

Lemma 2.11. *For a caterpillar graph G , $cn(G) = \max\{\eta, \sum_{i=1}^n d(u_i) - 2(n - 3)\}$ where η is the maximum number of times a positive integer appears as a degree of internal vertices of G .*

Proof. The degree sequence of the vertices of V_1 is known and if η is the maximum number of times a number appears as the degree of vertices in V_1 . Then the curling number of the caterpillar $G = \max\{\eta, d(u_i) - 2(n - 3)\}$. For both vertices u_1 and u_n , the values $d(u_1) - 1$ and $d(u_n) - 1$ represent the number of leafs attached to u_1, u_n respectively. Similarly, for the vertex $u_i, 2 \leq i \leq n - 1$, the values $d(u_i) - 2$

represents the number of leaves attached to the vertex u_i . Therefore the maximum number of leaves = $\sum_{i=1}^n d(u_i) - 2(n - 3)$. \square

Even though we can find out the compound curling number of specific caterpillar graph, calculation of the compound curling number of an arbitrary caterpillar graph G is very complex because of insufficient knowledge about the degree of the internal vertices of G .

3 Conclusion

In this paper, we have discussed the curling numbers of certain graph classes and their powers. There are several problems in this area which seem to be promising for further investigations. Some of the open problems we have identified during our study are the following.

Problem 3.1. Determine the curling number and the compound curling number of an arbitrary binary tree on n vertices.

Problem 3.2. Determine the curling number and the compound curling number of an n -ary tree on n vertices.

Problem 3.3. Determine the curling number and the compound curling number of an arbitrary tree on n vertices.

Problem 3.4. Determine the curling number and the compound curling number of an arbitrary caterpillar.

The concepts of curling number and compound curling number of certain graph powers and discussed certain properties of these new parameters for certain standard graphs. More problems regarding the curling number and compound curling number of certain other graph classes, graph operations, graph products and graph powers are still to be settled. All these facts highlights a wide scope for further studies in this area.

References

- [1] B. Chaffin, J. P. Linderman, N. J. A. Sloane, and A. R. Wilks, *On Curling Numbers of Integer Sequences*, arXiv:1212.6102 [math.CO], 2013.
- [2] J. A. Bondy and U. S. R. Murty, **Graph Theory**, Springer, 2008.
- [3] A. Brandstadt, V. B. Le and J. P. Spinard, **Graph Classes : A Survey**, SIAM, Philadelphia, 1999.
- [4] G. Chartrand and P. Zhang, **Chromatic Graph Theory**, CRC Press, Western Michigan University Kalamazoo, MI, U.S.A., 2009.

- [5] N. Deo, **Graph Theory with Applications to Engineering and Computer Science**, PHI Learning, 1974.
- [6] J. A. Gallian, *A Dynamic survey of Graph Labeling*, the electronic journal of combinatorics (DS-6), 2014.
- [7] F. Harary, **Graph Theory**, Addison-Wesley Publishing Company Inc, 1994.
- [8] J. Kok, N. Sudev , S. Chithra , *On Curling Number of Certain Graphs*, preprint,2015. arXiv Id :1506.00813[math.CO].
- [9] D. B. West, **Introduction to Graph Theory**, Pearson Education Asia, (2002).
- [10] E. W. Weisstein (2011). CRC Concise Encyclopedia of Mathematics, CRC press.